

# A RELATIONAL DESCRIPTION OF HIGHER COMMUTATORS IN MAL'CEV VARIETIES

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**ABSTRACT.** We give a relational description of higher commutator operators, which were introduced by Bulatov, in varieties with a Mal'cev term. Furthermore, we use this result to prove that for every algebra with a Mal'cev term there exists a largest clone on the same underlying set containing the Mal'cev operation and having the same congruence lattice and the same higher commutator operators as the original algebra. A local variant of this theorem is given.

## 1. INTRODUCTION

Two algebras are called *polynomially equivalent* if they have the same underlying set, and the same clone of all polynomial operations. One of the invariants to distinguish polynomially inequivalent algebras is the congruence lattice of the corresponding algebra, and the binary commutator operation  $[\cdot, \cdot]$  on this lattice (the theory describing this commutator have been developed in the 80's, and is described in the book by Freese and McKenzie [7]). In fact, from results of Idziak [8] and Bulatov [5], one can see that on the three-element set, every Mal'cev algebra is up to polynomial equivalence described by its congruence lattice, and the binary commutator operation. The analogue is not true for sets with at least four elements. But one can generalize the binary commutator operator to higher arities. These higher arity commutators have been introduced by Bulatov in [4]. From the description of polynomial clones on four-element set given in [6], one can obtain that four-element Mal'cev algebra is determined up to polynomial equivalence by its unary polynomials, congruence lattice, and higher commutator operators on this lattice. The higher commutators are defined by the following 'term-condition'.

**Definition 1.1** (Bulatov's higher commutator operators). Let  $\alpha_0, \dots, \alpha_{n-1}$ , and  $\gamma$  be congruences of some algebra  $\mathbf{A}$ . We say that  $\alpha_0, \dots, \alpha_{n-2}$  *centralize*  $\alpha_{n-1}$  *modulo*  $\gamma$  if for all tuples  $\mathbf{a}_i, \mathbf{b}_i, i = 0, \dots, n-1$ , and all terms  $t$  of  $\mathbf{A}$  such that

- (1)  $\mathbf{a}_i \neq \mathbf{b}_i$ , but the corresponding entries are congruent modulo  $\alpha_i$  for all  $i \in \{0, \dots, n-1\}$ , and
- (2)  $t(\mathbf{x}_0, \dots, \mathbf{x}_{n-2}, \mathbf{a}_{n-1}) \equiv_\gamma t(\mathbf{x}_0, \dots, \mathbf{x}_{n-2}, \mathbf{b}_{n-1})$  for all  $(\mathbf{x}_0, \dots, \mathbf{x}_{n-2}) \in (\{\mathbf{a}_0, \mathbf{b}_0\} \times \dots \times \{\mathbf{a}_{n-2}, \mathbf{b}_{n-2}\}) \setminus \{(\mathbf{b}_0, \dots, \mathbf{b}_{n-2})\}$ ,

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we have

$$t(\mathbf{b}_0, \dots, \mathbf{b}_{n-2}, \mathbf{a}_{n-1}) \equiv_{\gamma} t(\mathbf{b}_0, \dots, \mathbf{b}_{n-2}, \mathbf{b}_{n-1}).$$

The  $n$ -ary *commutator*  $[\alpha_0, \dots, \alpha_{n-1}]$  is then defined as the smallest congruence  $\gamma$  such that  $\alpha_0, \dots, \alpha_{n-2}$  centralize  $\alpha_{n-1}$  modulo  $\gamma$ .

One of the important notion that came from the higher commutators is a notion of super-nilpotence, an algebra is  $k$ -supernilpotent if it satisfies the commutator identity

$$\underbrace{[1, 1, \dots, 1]}_{k+1} = 0.$$

If an algebra is  $k$ -supernilpotent for some  $k$  we say shortly that is it supernilpotent. Though supernilpotence is in general stronger notion then nilpotence. However in groups both notions coincide; every  $k$ -nilpotent group is also  $k$ -supernilpotent. Supernilpotent algebras share some properties of nilpotent groups, in particular a Mal'cev algebra is supernilpotent if and only if it is product of prime-order supernilpotent algebras [1]. It has been shown in [2] that there are two expansions of the same group that are both 2-supernilpotent, but the join of their clones isn't. In this paper we establish additional properties to ensure that the join of two  $k$ -supernilpotent clones sharing the same Mal'cev operation is again  $k$ -supernilpotent.

To achieve that goal we give a description of higher commutators using a certain  $2^n$ -ary relation, denoted  $\Delta(\alpha_0, \dots, \alpha_{n-1})$  (see Definition 3.1). The similar relation denoted  $M_G$  have been also defined in [10]. The relation  $\Delta(\alpha_0, \dots, \alpha_{n-1})$  encodes the value of  $[\alpha_0, \dots, \alpha_{n-1}]$  as its *forks*—by a *fork* of a relation  $R$  at coordinate  $i$  we mean a pair  $(a, b)$  such that there exists  $\mathbf{c}, \mathbf{d} \in R$  with  $c_i = a$ ,  $d_i = b$ , and  $c_j = d_j$  for all  $j \neq i$ ; and we denote  $\psi_i(R)$ , the set of all forks of  $R$  at  $i$ . The description is then given by the following theorem.

**Theorem 1.2.** *If  $\mathbf{A}$  is a Mal'cev algebra, and  $\alpha_1, \dots, \alpha_n$  are congruences of  $\mathbf{A}$  then*

$$[\alpha_1, \dots, \alpha_n] = \psi_{2^n-1}(\Delta(\alpha_1, \dots, \alpha_n)).$$

Further we show that  $\Delta(\alpha_0, \dots, \alpha_{n-1})$  encodes not only the commutator  $[\alpha_0, \dots, \alpha_{n-1}]$  but also all smaller-arity commutators that can be obtained by omitting one or more of the congruences  $\alpha_i$ . We show that if we take the clone of all polymorphisms of the relation  $\Delta(\alpha_0, \dots, \alpha_{n-1})$  we get exactly the clone  $\mathcal{C}(\alpha_0, \dots, \alpha_{n-1})$  with the properties described in the following theorem, and consequently one can construct a largest clone with the same commutator operators as the original Mal'cev algebra.

**Theorem 1.3.** *Let  $\mathbf{A}$  be an algebra with Mal'cev term  $q$ , and let  $\alpha_0, \dots, \alpha_{n-1}$  be congruences of  $\mathbf{A}$ . Then there exists a largest clone  $\mathcal{C}(\alpha_0, \dots, \alpha_{n-1})$  on  $A$  containing  $q$  such that it preserves congruences  $\alpha_0, \dots, \alpha_{n-1}$ , and all commutators of the form  $[\alpha_{i_0}, \dots, \alpha_{i_{k-1}}]$  (where  $k \leq n$  and  $i_0 < \dots < i_{k-1} < n$ ) agree in  $\mathbf{A}$  and  $(A, \mathcal{C}(\alpha_0, \dots, \alpha_{n-1}))$ .*

**Corollary 1.4.** *Let  $\mathbf{A}$  be an algebra with a Mal'cev term  $q$ , then there exists a largest clone on  $A$  containing  $q$  such that the algebra corresponding to the clone has the same congruence lattice as  $\mathbf{A}$  and the same higher commutator operators as  $\mathbf{A}$ .*

*Proof of Corollary 1.4 given Theorem 1.3.* The largest such clone is the intersection of all clones  $\mathcal{C}(\alpha_1, \dots, \alpha_n)$  from Theorem 1.3 for all  $n$  and all congruences  $\alpha_1, \dots, \alpha_n$  of  $\mathbf{A}$ .  $\square$

The theory developed to prove Theorem 1.3 is strong enough to give relatively short proofs of several basic properties of higher commutators (usually referred as (HC1)–(HC8)) that have been established in [4], their proofs have been published in [1]. Our alternative proofs of some of these properties are given in the last section of this paper.

## 2. PRELIMINARIES AND NOTATION

Algebras are denoted by bold letters, the underlying set of an algebra is denoted by the same letter in italic,  $\text{Con } \mathbf{A}$  denotes the set of all congruences of an algebra  $\mathbf{A}$ ,  $\text{Clo } \mathbf{A}$  the set (clone) of all term operations of  $\mathbf{A}$ ,  $\text{Cg } X$  denotes the congruence generated by  $X$ ,  $\text{Sg } X$  denotes a subalgebra generated by  $X$ , and if  $\alpha$  is a congruence then we use the symbol  $a \equiv_{\alpha} b$  to denote  $(a, b) \in \alpha$ . Furthermore, if  $R$  is a relation, we use symbol  $\text{Pol } R$  to denote the set of all polymorphisms of  $R$ .

In the theory of binary commutator described in [7], it is usual to use the 4-ary relation  $\Delta_{\alpha, \beta}$  (we will denote the same relation  $\Delta(\alpha, \beta)$ ) whose elements are traditionally written as  $2 \times 2$  matrices. Similarly, when generalizing this concept to  $\Delta(\alpha_0, \dots, \alpha_{n-1})$  one should write elements of this relation as  $2 \times \dots \times 2$  ‘ $n$ -dimensional matrices’. We will write down these elements as tuples, and we introduce a rather technical notation to and speak about these elements easily. Nevertheless, it might be useful to imagine the elements of  $\Delta(\alpha_0, \dots, \alpha_{n-1})$  as  $n$ -dimensional hypercubes.

We denote tuples by bold letters. The  $i$ -th coordinate of tuple  $\mathbf{a}$  is denoted by either  $a_i$ , or  $\mathbf{a}_{(i)}$ . So,  $\mathbf{a} = (a_0, \dots, a_{n-1})$  and  $(a_0, \dots, a_{n-1})_{(i)} = a_i$ . In particular, we will use the symbol  $(a, b)_{(i)}$  to denote  $a$  if  $i = 0$ , or  $b$  if  $i = 1$ . All tuples will be numbered by consecutive increasing sequence of integers starting at 0. So every  $n$ -ary relation is a subset of  $A^{\{0, \dots, n-1\}} = A^n$ . In particular, the projection  $\pi_S(\mathbf{b})$  of  $n$ -tuple  $\mathbf{a}$  to a  $k$ -element set  $S$  is the tuple  $\mathbf{b}$  such that  $b_k = a_{i_k}$  where  $i_j$ ’s are elements of  $S$  such that  $i_0 < i_1 < \dots < i_{k-1}$ . We label the vertices of an  $n$ -dimensional hypercube by  $0, \dots, 2^n - 1$  in such a way that  $i$  and  $j$  are on the same edge if their binary expansion differs in exactly one digit. Whenever we refer to the  $i$ -th position of an index  $k$ , we mean the  $(i+1)$ -th digit from the right of the binary expansion of the corresponding number; we will denote this digit  $k_{(i)}$ . So,  $k = \sum_{i=1}^n 2^i k_{(i)}$ . Also, we will use the symbol  $\oplus$  as bitwise addition modulo two, and the symbol  $\wedge$  for bitwise multiplication (i.e.,  $12 \oplus 10 = 6$  and  $12 \wedge 10 = 8$ ).

We will also use a similar notation for tuples to one that is used for sets. Suppose  $f$  is a function defined on a totally ordered set  $S$ , and  $\phi(x)$  is a predicate. Then we denote

$$(f(k) \mid \phi(k))$$

the tuple  $(f(k_0), \dots, f(k_{n-1}))$  where  $k_i$ ’s are increasingly numbered elements of the set  $\{k \in S \mid \phi(k)\}$ . We will also use the same notation for arguments of functions, for example if we are in the same setting as above, and in addition  $f$  is a congruence valued function, then  $[f(k) \mid \phi(k)]$  denotes the commutator  $[f(k_0), \dots, f(k_{n-1})]$ , etc.

The last notation has a close connection to a simple lemma about forks of a relation. For any map  $e: \{0, \dots, n-1\} \rightarrow \{0, \dots, m-1\}$  and  $\mathbf{a} \in A^m$  the symbol  $\mathbf{a}^e$  denotes the  $n$ -tuple  $(a_{e(0)}, \dots, a_{e(n-1)})$ . Similarly, for a relation  $R \leq \mathbf{A}^m$ ,  $R^e$  denotes the relation  $\{\mathbf{a}^e \mid \mathbf{a} \in R\}$ .

**Lemma 2.1.** *Let  $\mathbf{A}$  be an algebra,  $R \leq \mathbf{A}^m$ ,  $S \leq \mathbf{A}^n$ ,  $e: \{0, \dots, n-1\} \rightarrow \{0, \dots, m-1\}$ ,  $R^e \leq S$ ,  $i < m$ . If there is unique  $j$  such that  $e(j) = i$  then  $\psi_i(R) \leq \psi_j(S)$ . In particular,*

- (i) *if  $R \leq S$  then  $\psi_i(R) \leq \psi_i(S)$ ;*
- (ii) *if  $S = \pi_I(R)$  then  $\psi_{e(j)}(R) \leq \psi_j(S)$  for all  $j < n$  where  $e$  is the unique strictly monotone map from  $\{1, \dots, n-1\}$  to  $I$ ;*
- (iii) *if  $e$  is bijective then  $\psi_{e(j)}(R) = \psi_j(R^e)$ .*

*Proof.* Suppose that  $(a, b) \in \psi_i(R)$ ; i.e., there are tuples  $\mathbf{a}, \mathbf{b} \in R$  such that  $a_i = a$ ,  $b_i = b$ , and  $a_k = b_k$  for all  $k \neq i$ . Then from  $R^e \leq S$  we know that  $\mathbf{a}^e, \mathbf{b}^e \in S$ . These tuples witness that  $(a, b) \in \psi_j(S)$ , because  $(\mathbf{a}^e)_{(j)} = a_i = a$ ,  $(\mathbf{b}^e)_{(j)} = b_i = b$ , and  $(\mathbf{a}^e)_{(k)} = a_{e(k)} = b_{e(k)} = (\mathbf{b}^e)_{(k)}$  for  $k \neq j$ .

The statements (i), (ii), and (iii) are special cases of the former. For (i) take  $m = n$ , and  $e$ , the identity mapping; for (ii) observe that  $R^e = \pi_I R$ , and  $e$  is injective, so the second condition for the unicity of  $j$  is satisfied. Finally, to prove (iii) suppose that  $e$  is a bijection on the set  $\{0, \dots, m-1\}$ . Then from the statement for  $S = R^e$  we get that

$$\psi_{e(j)}(R) \leq \psi_j(R^e).$$

For the other inclusion substitute  $e$  with  $e^{-1}$ ,  $R$  with  $R^e$ , and  $j$  with  $e(j)$ . □

We recall two simple well-known lemmata for Mal'cev algebras.

**Lemma 2.2.** *Let  $\mathbf{A}$  be a Mal'cev algebra. Then any binary reflexive compatible relation on  $\mathbf{A}$  is a congruence.* □

**Lemma 2.3.** *Let  $\mathbf{A}$  be a Mal'cev algebra, and let  $R$  be  $n$ -ary compatible relation on  $\mathbf{A}$ . If  $(a, b) \in \psi_i(R)$ , and  $(c_0, \dots, c_{i-1}, a, c_{i+1}, \dots, c_{n-1}) \in R$  then*

$$(c_0, \dots, c_{i-1}, b, c_{i+1}, \dots, c_{n-1}) \in R.$$

*Proof.* Without loss of generality suppose that  $i = 0$ . Let  $q$  be a Mal'cev term of  $\mathbf{A}$ , and let  $(a, u_1, \dots, u_{n-1})$  and  $(b, u_1, \dots, u_{n-1})$  be witnesses for  $(a, b) \in \psi_0(R)$ . Then

$$q \left( \begin{array}{ccc} b & a & a \\ u_1 & u_1 & c_1 \\ \vdots & & \vdots \\ u_{n-1} & u_{n-1} & c_{n-1} \end{array} \right) = \left( \begin{array}{c} b \\ c_1 \\ \vdots \\ c_{n-1} \end{array} \right) \in R,$$

since we know that  $R$  is compatible with the Mal'cev operation  $q$ . □

### 3. DESCRIPTION OF HIGHER COMMUTATORS

**Definition 3.1.** Let  $\mathbf{A}$  be an algebra, and  $\alpha_0, \dots, \alpha_{n-1} \in \text{Con } \mathbf{A}$ . First, for each congruence  $\alpha_i$  choose one direction in  $n$ -dimensional space. We define the relation  $\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1})$  as the  $2^n$ -ary relation generated by tuples that are constant on two opposing  $(n-1)$ -dimensional hyperfaces of the hypercube orthogonal to the direction corresponding to  $\alpha_i$  and these constants are  $\alpha_i$  congruent.

The generating tuples will be denoted  $\mathbf{c}_i^n(a, b)$ . To write it down formally, we will use the map  $d_{i,n}: \{0, \dots, 2^n - 1\} \rightarrow \{0, 1\}$  defined by  $k \mapsto k_{(i)}$ ; so  $\mathbf{c}_i^n(a, b) = (a, b)^{d_{i,n}} = ((a, b)_{(k_{(i)})} \mid k < 2^n)$ . Then  $\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1})$  is defined as

$$\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1}) := \text{Sg} \{ \mathbf{c}_i^n(a, b) \mid i < n, a \equiv_{\alpha_i} b \},$$

or equivalently,

$$\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1}) := \bigvee_{i < n} \alpha_i^{d_{i,n}}.$$

For the trivial case when  $n = 0$ , we put  $\Delta_{\mathbf{A}}() := A$ .

If the algebra is clear from the context, we will write just  $\Delta(\alpha_0, \dots, \alpha_{n-1})$  instead of  $\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1})$ , and if  $\mathcal{C}$  is a clone on the set  $A$ , we will write  $\Delta_{\mathcal{C}}(\alpha_0, \dots, \alpha_{n-1})$  instead of  $\Delta_{(A, \mathcal{C})}(\alpha_0, \dots, \alpha_{n-1})$ .

*Example.* We will describe generators of  $\Delta(\alpha_0, \alpha_1, \alpha_2)$  for three congruences  $\alpha_0, \alpha_1, \alpha_2$  of an algebra  $\mathbf{A}$ . The elements of  $\Delta(\alpha_0, \alpha_1, \alpha_2)$  are indexed by vertices of a three-dimensional hypercube. The generators are tuples of one of the following forms  $(a, b, a, b, a, b, a, b)$ , where  $a \equiv_{\alpha_0} b$ ,  $(a, a, b, b, a, a, b, b)$  where  $a \equiv_{\alpha_1} b$ , and  $(a, a, a, a, b, b, b, b)$  where  $a \equiv_{\alpha_2} b$ . Their graphical representation is given in Figure 1.

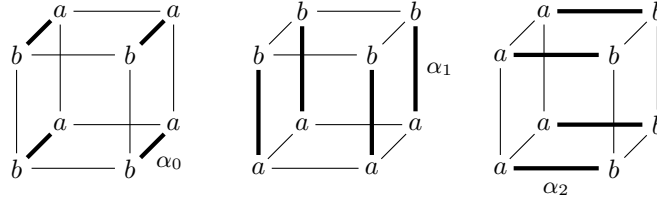


FIGURE 1. Generators of  $\Delta(\alpha_0, \alpha_1, \alpha_2)$

Before we get to the proof of Theorem 1.2, we will describe some basic properties of the relation  $\Delta(\alpha_0, \dots, \alpha_{n-1})$ . The first lemma gives a term description of  $\Delta(\alpha_0, \dots, \alpha_{n-1})$ . This description gives a clear connection of  $\Delta(\alpha_1, \dots, \alpha_n)$  and the term condition.

**Lemma 3.2.** *For every algebra  $\mathbf{A}$ , and congruences  $\alpha_i \in \text{Con } A$ ,  $i < n$ ,*

$$\Delta(\alpha_0, \dots, \alpha_{n-1}) = \{ (t((\mathbf{a}_0, \mathbf{b}_0)_{(k_{(0)})}, \dots, (\mathbf{a}_{n-1}, \mathbf{b}_{n-1})_{(k_{(n-1)})}) \mid k < 2^n) \mid \\ \forall i < n : m_i \in \mathbb{N}_0, \mathbf{a}_i, \mathbf{b}_i \in A^{m_i}, \mathbf{a}_i \equiv_{\alpha_i} \mathbf{b}_i; t \in \text{Clo}_{\Sigma_{m_i}} \mathbf{A} \}.$$

*Proof.* The relation  $\Delta(\alpha_0, \dots, \alpha_{n-1})$  is generated by tuples  $c_i^n(a, b)$  for  $a \equiv_{\alpha_i} b$ . So,  $\Delta(\alpha_0, \dots, \alpha_{n-1})$  is the set of all tuples of the form

$$t(\mathbf{c}_{i_0}^n(a_0, b_0), \dots, \mathbf{c}_{i_{k-1}}^n(a_{k-1}, b_{k-1}))$$

where  $t \in \text{Clo}_k \mathbf{A}$ , and for all  $j < k$  we have  $i_j < n$ ,  $a_j \equiv_{\alpha_{i_j}} b_j$  modulo  $\alpha_{i_j}$ . The description in the statement of the lemma can be obtained from this by grouping together  $\mathbf{c}_{i_j}^n(a_j, b_j)$ 's with the same index  $i_j$ , and applying the term  $t$  coordinate-wise.  $\square$

*Example.* In the ternary commutator case, the lemma tells that

$$\Delta(\alpha_0, \dots, \alpha_{n-1}) = \{ (t(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2), t(\mathbf{b}_0, \mathbf{a}_1, \mathbf{a}_2), t(\mathbf{a}_0, \mathbf{b}_1, \mathbf{a}_2), t(\mathbf{b}_0, \mathbf{b}_1, \mathbf{a}_2), \\ t(\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_2), t(\mathbf{b}_0, \mathbf{a}_1, \mathbf{b}_2), t(\mathbf{a}_0, \mathbf{b}_1, \mathbf{b}_2), t(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2)) \mid \\ m_0, m_1, m_2 \in \mathbb{N}_0, t \in \text{Clo}_{m_0+m_1+m_2} \mathbf{A}, \forall i < 3 : \mathbf{a}_i, \mathbf{b}_i \in A^{m_i}, \mathbf{a}_i \equiv_{\alpha_i} \mathbf{b}_i \}.$$

The graphical representation of a typical element of this relation is given in Figure 2.

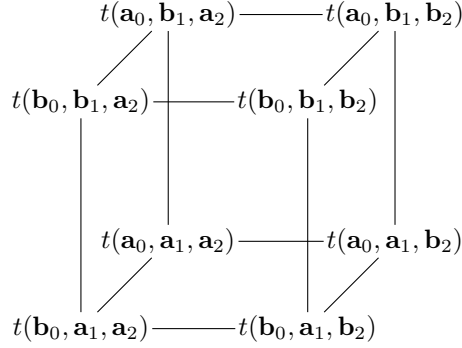


FIGURE 2.

**Lemma 3.3.** *Let  $\mathbf{A}$  be an algebra, and  $\alpha_0, \dots, \alpha_{n-1}, \gamma \in \text{Con } \mathbf{A}$ . Then  $\alpha_0, \dots, \alpha_{n-2}$  centralize  $\alpha_{n-1}$  modulo  $\gamma$  if for every tuple*

$$(a_0, \dots, a_{2^n-1}) \in \Delta(\alpha_0, \dots, \alpha_{n-1})$$

*such that  $a_i \equiv_\gamma a_{i+2^{n-1}}$  for all  $i < 2^{n-1} - 1$  we have also  $a_{2^{n-1}-1} \equiv_\gamma a_{2^n-1}$ .  $\square$*

**Lemma 3.4.** *Let  $\mathbf{A}$  be an algebra with congruences  $\alpha_0, \dots, \alpha_n$ , let  $i < n$ , and  $d \in \{0, 1\}$ . Then*

- (i)  $\pi_{\{k(i)=d\}}(\Delta(\alpha_0, \dots, \alpha_{n-1})) = \Delta(\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n-1});$
- (ii) *if  $\pi_{\{k(i)=0\}}\mathbf{a} = \pi_{\{k(i)=1\}}\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n-1})$  then*  

$$\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-1});$$
- (iii) *if  $\sigma_i: k \mapsto k \oplus 2^i$  then*

$$\Delta(\alpha_0, \dots, \alpha_{n-1})^{\sigma_i} = \Delta(\alpha_0, \dots, \alpha_{n-1}).$$

*Furthermore, if  $\mathbf{A}$  is a Mal'cev algebra then the binary relation*

$$\delta = \{((a_k | k(i) = 0), (a_k | k(i) = 1)) | \mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-1})\}$$

*is a congruence on  $\Delta(\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n)$ .*

*Proof.* Proofs of these facts are based on the simple idea to test the validity of the statement on the generators of the depicted algebras, i.e., tuples of the form  $\mathbf{c}_{i,n}(a, b)$ . In detail, to prove (i) one has to observe that

$$\begin{aligned} \pi_{\{k(i)=d\}}\{\mathbf{c}_{j,n}(a, b) | j < n, a \equiv_{\alpha_j} b\} &= \{\mathbf{c}_{j,n-1}(a, b) | j < i, a \equiv_{\alpha_j} b\} \\ &\cup \{(a, \dots, a) | a \in A\} \cup \{\mathbf{c}_{j-1,n-1}(a, b) | i < j < n, a \equiv_{\alpha_j} b\}. \end{aligned}$$

Hence the relation generated by the left hand side is the same as the relation generated by the right hand side, which gives the desired equality.

For (ii), first observe that

$$\{\mathbf{a} | \pi_{\{k(i)=0\}}\mathbf{a} = \pi_{\{k(i)=1\}}\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n-1})\}$$

is the  $2^n$ -ary relation generated by tuples  $\mathbf{a}$  such that  $\pi_{\{k(i)=0\}}\mathbf{a} = \pi_{\{k(i)=1\}}\mathbf{a} = \mathbf{c}_{j,n-1}(a, b)$ ,  $a \equiv b$  modulo  $\alpha_j$ , or  $\alpha_{j+1}$  when  $j < i$ , or  $j \geq i$  resp. Second, if  $\mathbf{a}$  is such a tuple then  $\mathbf{a} = \mathbf{c}_{j,n}(a, b)$  if  $j < i$ , or  $\mathbf{a} = \mathbf{c}_{j+1,n}(a, b)$  if  $j \geq i$ . In either case,  $\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-1})$  for  $a, b$  that are congruent modulo the corresponding congruence.

The item (iii) is a consequence of the fact that

$$(\mathbf{c}_{j,n}(a, b))^{\sigma_i} = \begin{cases} \mathbf{c}_{j,n}(b, a) & \text{if } j = i, \text{ or} \\ \mathbf{c}_{j,n}(a, b) & \text{otherwise.} \end{cases}$$

Hence,  $(\alpha_j^{d_{j,n}})^{\sigma_i} = \alpha_j^{d_{j,n}}$  from the symmetry of congruences, and from the definition of  $\Delta(\alpha_0, \dots, \alpha_{n-1})$ , we get that  $\Delta(\alpha_0, \dots, \alpha_{n-1})^{\sigma_i} = \Delta(\alpha_0, \dots, \alpha_{n-1})$ .

From items (i)–(iii) we already know that the binary relation  $\delta$  is a reflexive symmetric binary relation on  $\Delta(\alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{n-1})$ . The rest follows from Lemma 2.2.  $\square$

**Lemma 3.5.** *Let  $\mathbf{A}$  be a Mal'cev algebra with congruences  $\alpha_0, \dots, \alpha_{n-1}$ . Then  $\psi_i(\Delta(\alpha_0, \dots, \alpha_{n-1})) = \psi_j(\Delta(\alpha_0, \dots, \alpha_{n-1}))$  for all  $i, j < 2^n$ .*

*Proof.* For simplicity let  $\psi_k = \psi_k(\Delta(\alpha_0, \dots, \alpha_{n-1}))$ . If  $\sigma_i$  is a permutation on  $\{0, \dots, 2^n - 1\}$  defined by  $\sigma_i(k) = k \oplus 2^i$  then from Lemma 3.4.iii, we know that  $\Delta(\alpha_0, \dots, \alpha_{n-1}) = \Delta(\alpha_0, \dots, \alpha_{n-1})^{\sigma_i}$ . This gives, by Lemma 2.1, that  $\psi_k = \psi_{\sigma_i(k)}$  for all  $i < n$ . But, if  $i_0, \dots, i_{m-1}$  are all the indices  $i$  such that  $k_{(i)} \neq j_{(i)}$  then  $j = \sigma_{i_0} \circ \dots \circ \sigma_{i_{m-1}}(k)$ , and consequently  $\psi_k = \psi_j$  for all  $j, k$ .  $\square$

Instead of proving Theorem 1.2 directly, we will prove the following refinement. The theorem is then given by equivalence of (1) and (4).

**Proposition 3.6.** *If  $\mathbf{A}$  is a Mal'cev algebra,  $\alpha_0, \dots, \alpha_{n-1} \in \text{Con } \mathbf{A}$ , and  $a, b \in \mathbf{A}$ ; then the following is equivalent*

- (1)  $(a, b) \in \psi_{2^{n-1}-1}(\Delta(\alpha_0, \dots, \alpha_{n-1}))$ ;
- (2)  $(a, \dots, a, b) \in \Delta(\alpha_0, \dots, \alpha_{n-1})$ ;
- (3) *there exists  $c_0, \dots, c_{2^{n-1}-2}$  such that*

$$(c_0, \dots, c_{2^{n-1}-2}, a, c_0, \dots, c_{2^{n-1}-2}, b) \in \Delta(\alpha_0, \dots, \alpha_{n-1}).$$

- (4)  $a \equiv_{[\alpha_0, \dots, \alpha_{n-1}]} b$ ;

*Proof.* The implication (1)  $\rightarrow$  (2) is direct corollary of Lemma 2.3, given that  $\Delta(\alpha_0, \dots, \alpha_{n-1})$  contains all constant tuples, especially  $(a, \dots, a)$ . (2)  $\rightarrow$  (3) is trivial. The implication (3)  $\rightarrow$  (4) is given by direct use of Lemma 3.3.

The last to prove is (4)  $\rightarrow$  (1), in other words that

$$(3.1) \quad [\alpha_0, \dots, \alpha_{n-1}] \leq \psi_{2^n-1}(\Delta(\alpha_0, \dots, \alpha_{n-1})).$$

Let  $\psi = \psi_{2^n-1}(\Delta(\alpha_0, \dots, \alpha_{n-1}))$ ; from Lemma 3.5 we know that

$$\psi = \psi_{2^n-1}(\Delta(\alpha_0, \dots, \alpha_{n-1})) = \psi_j(\Delta(\alpha_0, \dots, \alpha_{n-1}))$$

for all  $j < 2^n$ , so we do not have to distinguish between forks at different coordinates. To prove (3.1), it is enough to prove that  $\alpha_0, \dots, \alpha_{n-2}$  centralize  $\alpha_{n-1}$  modulo  $\psi$ . For that we will use Lemma 3.3. Suppose that

$$(a_0, \dots, a_{2^{n-1}-1}, b_0, \dots, b_{2^{n-1}-1}) \in \Delta(\alpha_0, \dots, \alpha_{n-1}),$$

and  $a_i \equiv_{\psi} b_i$  for all  $i < 2^{n-1} - 1$ . By repeated use of Lemma 2.3 we can replace  $b_0, \dots, b_{2^{n-1}-2}$  by the respective  $a_i$ 's. Hence,

$$(a_0, \dots, a_{2^{n-1}-2}, a_{2^{n-1}-1}, a_0, \dots, a_{2^{n-1}-2}, b_{2^{n-1}-1}) \in \Delta(\alpha_0, \dots, \alpha_{n-1}).$$

In addition, we know from Lemma 3.4.ii that

$$(a_0, \dots, a_{2^{n-1}-2}, a_{2^{n-1}-1}, a_0, \dots, a_{2^{n-1}-2}, a_{2^{n-1}-1}) \in \Delta(\alpha_0, \dots, \alpha_{n-1}).$$

So,  $a_{2^{n-1}-1} \equiv_{\psi} b_{2^{n-1}-1}$  which concludes the proof that  $\alpha_0, \dots, \alpha_{n-2}$  centralize  $\alpha_{n-1}$  modulo  $\psi$ .  $\square$

In the last proposition some parts have been already known. The proposition (in the case of Mal'cev algebras) generalize Theorem 4.9 of [7] which gives equivalence of (2), (3), and (4) for a binary commutator in the congruence modular varieties. The omitted equivalence of (1) and (3) in the binary case can be easily derived from the known fact that  $\Delta(\alpha_0, \alpha_1)$  is a congruence on rows. Furthermore, for the higher commutators, the implication (3)  $\rightarrow$  (4) is known in the variety of groups from [10, Lemma 3.3]; and if all  $\alpha_i$ 's are principal congruences, the equivalence of (2) and (4) is given by [1, Lemma 6.13] (via an easy translation similar to Lemma 3.2).

#### 4. STRONG CUBE TERMS, AND CLONES OF OPERATIONS PRESERVING COMMUTATORS

We will use terms that are a generalization of a Mal'cev term. These terms will play similar role as a difference term in the case of binary commutator. Term  $q_n$  is *strong  $n$ -cube term* if it satisfies

$$q_n(x_0, \dots, x_{2^n-2}) = x_{2^n-1}$$

whenever there is  $i < n$  such that for all  $k, l < 2^n$  which differs only in  $(i+1)$ -th digit we have  $x_k = x_l$ . This gives a set of  $n$  identities, each with  $2^{n-1}$  variables. The two identities for strong 2-cube term are

$$q_2(x, y, x) = y \quad \text{and} \quad q_2(x, x, y) = y.$$

So, the term  $q_2(y, x, z)$  is a Mal'cev term and if  $q$  is a Mal'cev term then  $q(y, x, z)$  is a strong 2-cube term. The three identities for strong 3-cube term are

$$\begin{aligned} q_3(x, y, z, w, x, y, z) &= w \\ q_3(x, y, x, y, z, w, z) &= w \\ q_3(x, x, y, y, z, z, w) &= w. \end{aligned}$$

Strong cube terms are stricter version of cube terms introduced in [3]—every strong  $n$ -cube term is an  $n$ -cube term.

**Lemma 4.1.** *Let  $n \geq 2$ . Then an algebra has a strong  $n$ -cube term if and only if it has a Mal'cev term.*

*Proof.* As written in the above paragraphs, one can define a strong 2-cube term from a Mal'cev term and vice-versa. Observe that if  $q_n$  is strong  $n$ -cube term and  $q_2$  is a strong 2 cube term then the term

$$q_{n+1}(x_0, \dots, x_{2^{n+1}-1}) = q_2(q_n(x_0, \dots, x_{2^n-2}), x_{2^n-1}, q_n(x_{2^n}, \dots, x_{2^{n+1}-2})).$$

is a strong  $(n+1)$ -cube term. So, we can define strong  $n$ -cube term for any  $n$  by the above recursion. Other way, if  $q_n$  is a strong  $n$ -cube term then  $q(x, y, z) = q_n(y, \dots, y, y, x, z)$  is a Mal'cev term.  $\square$

The following lemma is the key for proving Theorem 1.3 and a lot of properties of higher commutators in Mal'cev varieties.

**Lemma 4.2.** *Let  $\mathbf{A}$  be algebra with strong  $n$ -cube term  $q_n$ ,  $\alpha_1, \dots, \alpha_n \in \text{Con } \mathbf{A}$ . Then  $\mathbf{a} \in \Delta(\alpha_1, \dots, \alpha_n)$  if and only if  $(a_k \mid k_{(j)} = 0) \in \Delta(\alpha_i \mid i \neq j)$  for each  $j$ , and  $q_n(a_0, \dots, a_{2^n-2}) \equiv_{[\alpha_1, \dots, \alpha_n]} a_{2^n-1}$ .*



*Proof.* We will prove the lemma in two steps.

*Claim 1.* If  $(a_k \mid k_{(j)} = 0) \in \Delta(\alpha_i \mid i \neq j)$  for each  $j$ , and  $q_n(a_0, \dots, a_{2^n-2}) = a_{2^n-1}$ , then  $\mathbf{a} \in \Delta(\alpha_1, \dots, \alpha_{n-1})$ ;

Recall that  $\wedge$  denotes bitwise multiplication, i.e.  $k \wedge l$  is the number that has digit 1 on some position if and only if both  $k$  and  $l$  has 1 on the corresponding position. For  $k < 2^n$  let  $m_k: \{0, \dots, 2^n - 1\} \rightarrow \{0, \dots, 2^n - 1\}$ ,  $l \mapsto l \wedge k$ . So,

$$\mathbf{b}^{m_k} = (b_{0 \wedge k}, \dots, b_{(2^{n-1}-1) \wedge k}).$$

From Lemma 3.4.(ii), we know that if  $k$  has only one 0 on the position  $i$ , i.e.,  $k = 2^n - 1 - 2^i$ , and  $(b_k : k_{(i)} = 0) \in \Delta(\alpha_j \mid j \neq i)$  then  $\mathbf{b}^{m_k} \in \Delta(\alpha_0, \dots, \alpha_{n-1})$ . From Lemma 3.4 we also know that the assumption is satisfied for all tuples  $\mathbf{b} \in \Delta(\alpha_0, \dots, \alpha_{n-1})$ .

Now we can show by induction on the number of 0's in the binary expansion of  $k$  that  $\mathbf{a}^{m_k} \in \Delta(\alpha_0, \dots, \alpha_{n-1})$  for all  $k < 2^n - 1$ . Both the first and the inductive step is given by the previous paragraph, and the following observation: if  $k_{(i)} = 0$  then  $\mathbf{a}^{m_k} = (\mathbf{a}^{m_{k+2^i}})^{m_{(2^n-1)-2^i}}$ .

Next we show that

$$(4.1) \quad q_n^{\Delta(\alpha_0, \dots, \alpha_{n-1})}(\mathbf{a}^{m_0}, \dots, \mathbf{a}^{m_{2^n-2}}) = (a_0, \dots, a_{2^n-2}, q_n^{\mathbf{A}}(a_0, \dots, a_{2^n-2})),$$

or, in other words,

$$q_n \begin{pmatrix} a_{0 \wedge 0} & \dots & a_{0 \wedge (2^{n-1}-2)} \\ \vdots & & \vdots \\ a_{(2^n-2) \wedge 0} & \dots & a_{(2^n-2) \wedge (2^{n-1}-2)} \\ a_{(2^n-1) \wedge 0} & \dots & a_{(2^n-1) \wedge (2^{n-1}-2)} \end{pmatrix} = \begin{pmatrix} a_0 \\ \vdots \\ a_{2^n-2} \\ q_n(a_0, \dots, a_{2^n-2}) \end{pmatrix}$$

The result is obvious for the last coordinate since the last row is  $(a_0, \dots, a_{2^n-2})$ . For the  $l$ -th row ( $l < 2^n - 1$ ), we want to show that

$$q_n(a_{l \wedge 0}, \dots, a_{l \wedge (2^n-2)}) = a_{l \wedge (2^n-1)}.$$

Since  $l < 2^{n-1} - 1$ , it has 0 on some position  $i$ . The above equality then follows from the  $i$ -th identity for a strong cube term. Finally, since the relation  $\Delta(\alpha_0, \dots, \alpha_{n-1})$  is compatible with  $q_n$  we know that the right hand side of (4.1) is in  $\Delta(\alpha_0, \dots, \alpha_{n-1})$ .

*Claim 2.* If  $\mathbf{a} \in \Delta(\alpha_1, \dots, \alpha_n)$  then  $q_n(a_0, \dots, a_{2^n-2}) \equiv_{[\alpha_1, \dots, \alpha_n]} a_{2^n-1}$ .

If  $(a_0, \dots, a_{2^n-1}) \in \Delta(\alpha_1, \dots, \alpha_n)$  then from Lemma 3.4.(i) we know that the tuple  $(a_0, \dots, a_{2^n-2}, q_n(a_0, \dots, a_{2^n-2}))$  satisfies the prerequisites of the first claim. Hence, we know that

$$(a_0, \dots, a_{2^n-2}, q_n(a_0, \dots, a_{2^n-2})) \in \Delta(\alpha_1, \dots, \alpha_n)$$

which shows that

$$(a_{2^n-1}, q_n(a_0, \dots, a_{2^n-2})) \in \psi_{2^n-1}(\Delta(\alpha_1, \dots, \alpha_n)) = [\alpha_1, \dots, \alpha_n].$$

Finally, we get to the statement of this lemma. The ‘only if’ part is Claim 2 together with Lemma 3.4.(i). For the ‘if’ part, if  $(a_k \mid k_{(j)}=0) \in \Delta(\alpha_i \mid i \neq j)$  we know from Claim 1 that  $(a_0, \dots, a_{2^n-2}, q_n(a_0, \dots, a_{2^n-2})) \in \Delta(\alpha_0, \dots, \alpha_{n-1})$ . Also, from the last condition and Theorem 1.2, we know that  $q_n(a_0, \dots, a_{2^n-2})$  and  $a_{2^n-1}$  are congruent modulo  $\psi_{2^n-1}(\Delta(\alpha_0, \dots, \alpha_{n-1}))$ . So,  $\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-1})$  from Lemma 2.3.  $\square$

In the rest of this chapter we use symbol  $[\alpha_0, \dots, \alpha_{n-1}]_{\mathbf{X}}$  to denote the commutator  $[\alpha_0, \dots, \alpha_{n-1}]$  computed in the algebra  $\mathbf{X}$ .

**Corollary 4.3.** *Let  $\mathbf{A}, \mathbf{B}$  be algebras that share a Mal'cev term  $q$ , and congruences  $\alpha_0, \dots, \alpha_{n-1}$ . Then  $[\alpha_i | i \in I]_{\mathbf{A}} = [\alpha_i | i \in I]_{\mathbf{B}}$  for all  $I \subseteq \{0, \dots, n-1\}$  if and only if  $\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1}) = \Delta_{\mathbf{B}}(\alpha_0, \dots, \alpha_{n-1})$ .*

*Proof.* We will prove the corollary by induction on  $n$ . The case  $n = 1$  is trivial; for the induction step suppose that for all congruences  $\beta_0, \dots, \beta_{n-1} \in \text{Con } \mathbf{A} \cap \text{Con } \mathbf{B}$  such that the commutators  $[\beta_{i_0}, \dots, \beta_{i_{k-1}}]$  agree in  $\mathbf{A}$  and  $\mathbf{B}$  for all  $k < n$  and  $i_0, \dots, i_{k-1}$  pairwise distinct elements from  $\{0, \dots, n-1\}$ , we have  $\Delta_{\mathbf{A}}(\beta_{i_0}, \dots, \beta_{i_{k-1}}) = \Delta_{\mathbf{B}}(\beta_{i_0}, \dots, \beta_{i_{k-1}})$ . In particular, we have  $\Delta_{\mathbf{A}}(\alpha_i | i \neq j) = \Delta_{\mathbf{B}}(\alpha_i | i \neq j)$  for all  $j < n$ . Let  $q_n$  be a common strong  $n$ -cube term of  $\mathbf{A}$  and  $\mathbf{B}$ . From Lemma 4.2 we know that  $\mathbf{a} \in \Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1})$  if and only if

$$(4.2) \quad (a_i | k_{(j)} = 0) \in \Delta_{\mathbf{A}}(\alpha_i | i \neq j) \text{ for all } j < n, \text{ and}$$

$$a_{2^n-1} \equiv_{[\alpha_0, \dots, \alpha_{n-1}]_{\mathbf{A}}} q_n(a_0, \dots, a_{2^n-2});$$

and the similar for  $\mathbf{B}$ . The condition (4.2) is identical for  $\mathbf{A}$  and  $\mathbf{B}$  since both the commutators  $[\alpha_0, \dots, \alpha_{n-1}]$  and the relations  $\Delta(\alpha_i | i \neq j)$  agree in  $\mathbf{A}$  and  $\mathbf{B}$ , so  $\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1}) = \Delta_{\mathbf{B}}(\alpha_0, \dots, \alpha_{n-1})$ .

For the ‘only if’ part, suppose that  $\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1}) = \Delta_{\mathbf{B}}(\alpha_0, \dots, \alpha_{n-1})$ . Hence, from Lemma 3.4.(i) we know that

$$\Delta_{\mathbf{A}}(\alpha_i | i \in I) = \Delta_{\mathbf{B}}(\alpha_i | i \in I)$$

for all  $I \subseteq \{0, \dots, n-1\}$ ; and consequently from Theorem 1.2,  $[\alpha_i | i \in I]_{\mathbf{A}} = [\alpha_i | i \in I]_{\mathbf{B}}$ .  $\square$

Finally we can get to the proof of the Theorem 1.3. We restate the theorem once again.

**Theorem 1.3.** *Let  $\mathbf{A}$  be an algebra with Mal'cev term  $q$ , and  $\alpha_0, \dots, \alpha_{n-1}$  be congruences of  $\mathbf{A}$ . Then there exists a largest clone  $\mathcal{C}$  on  $A$  containing  $q$  such that it preserves congruences  $\alpha_0, \dots, \alpha_{n-1}$ , and all commutators of the form  $[\alpha_i | i \in I]$  where  $I \subseteq \{1, \dots, n\}$  agree in  $\mathbf{A}$  and  $(A, \mathcal{C})$ .*

*Proof.* Let  $\mathcal{C} = \text{Pol}(\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1}))$ . We will show that this clone has the required properties. Obviously  $\mathcal{C} \geq \text{Clo } \mathbf{A}$  which implies that  $q \in \mathcal{C}$  and  $\Delta_{\mathcal{C}}(\alpha_0, \dots, \alpha_{n-1}) \geq \Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1})$ . But since  $\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1})$  is compatible with  $\mathcal{C}$  and  $\Delta_{\mathcal{C}}(\alpha_0, \dots, \alpha_{n-1})$  is the smallest compatible relation containing  $\mathbf{c}_i^n(a, b)$  for all  $a \equiv_{\alpha_i} b$  and  $i < n$ , we have  $\Delta_{\mathcal{C}}(\alpha_0, \dots, \alpha_{n-1}) = \Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1})$ .

From Corollary 4.3, we know that  $\mathcal{C}$  satisfies the specified property. The rest is to prove that  $\mathcal{C}$  is the largest such clone. Let  $\mathcal{B}$  be another clone with the property. Then again from the same corollary,  $\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1}) = \Delta_{\mathcal{B}}(\alpha_0, \dots, \alpha_{n-1})$ , and consequently  $\mathcal{B} \subseteq \text{Pol}(\Delta_{\mathbf{A}}(\alpha_0, \dots, \alpha_{n-1})) = \mathcal{C}$ .  $\square$

## 5. PROOFS OF BASIC PROPERTIES OF HIGHER COMMUTATORS

In this chapter, we will present alternative proofs of basic properties of higher commutators formulated in [4, 1]. For an arbitrary algebra  $\mathbf{A}$  and its congruences  $\alpha_0, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_{n-1}, \gamma$ , and  $\eta$  the following is satisfied

$$(\text{HC1}) \quad [\alpha_0, \dots, \alpha_n] \leq \alpha_0 \wedge \dots \wedge \alpha_{n-1};$$

$$(\text{HC2}) \quad \text{if } \alpha_i \leq \beta_i \text{ for all } i \text{ then } [\alpha_0, \dots, \alpha_n] \leq [\beta_0, \dots, \beta_{n-1}];$$

(HC3)  $[\alpha_0, \dots, \alpha_n] \leq [\alpha_1, \dots, \alpha_{n-1}]$ .

Furthermore, if  $\mathbf{A}$  is a Mal'cev algebra then

(HC4) if  $\sigma$  is a permutation on the set  $\{0, \dots, n-1\}$  then

$$[\alpha_0, \dots, \alpha_{n-1}] = [\alpha_{\sigma(0)}, \dots, \alpha_{\sigma(n-1)}];$$

(HC5) congruences  $\alpha_0, \dots, \alpha_{n-2}$  centralize  $\alpha_{n-1}$  modulo  $\gamma$  if and only if

$$[\alpha_0, \dots, \alpha_{n-1}] \leq \gamma;$$

(HC6) if  $\eta \leq \alpha_0, \dots, \alpha_{n-1}$  then

$$[\alpha_0/\eta, \dots, \alpha_{n-1}/\eta]_{\mathbf{A}/\eta} = ([\alpha_0, \dots, \alpha_{n-1}]_{\mathbf{A}} \vee \eta)/\eta;$$

(HC7) if  $I$  is a non-empty set, and  $\rho_i$  are congruences of  $\mathbf{A}$  for all  $i \in I$  then

$$\bigvee_{i \in I} [\alpha_0, \dots, \alpha_{n-2}, \rho_i] = [\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in I} \rho_i];$$

(HC8) if  $i = 1, \dots, n-1$  then  $[[\alpha_0, \dots, \alpha_{i-1}], \alpha_i, \dots, \alpha_{n-1}] \leq [\alpha_0, \dots, \alpha_{n-1}]$ .

Although properties (HC1–3) can be derived directly from Theorem 1.2 and the definition of  $\Delta(\alpha_0, \dots, \alpha_{n-1})$ , we will omit these proofs because such proofs would work only for Mal'cev algebras; the general case have been proved in [9, Proposition 1.3]. We will prove properties (HC4), (HC5), (HC7), and (HC8)—the last property (HC6) is a corollary of (HC5).

**Proposition 5.1** (HC4, [1, Proposition 6.1]). *Let  $\mathbf{A}$  be a Mal'cev algebra and let  $\alpha_0, \dots, \alpha_{n-1} \in \text{Con } \mathbf{A}$ . Then  $[\alpha_0, \dots, \alpha_{n-1}] = [\alpha_{\sigma(0)}, \dots, \alpha_{\sigma(n-1)}]$  for each permutation  $\sigma$  of  $\{0, \dots, n-1\}$ .*

*Proof.* We claim that the relations  $\Delta(\alpha_{\sigma(0)}, \dots, \alpha_{\sigma(n-1)})$  and  $\Delta(\alpha_0, \dots, \alpha_{n-1})$  are identical up to permuting coordinates; precisely

$$\Delta(\alpha_{\sigma(0)}, \dots, \alpha_{\sigma(n-1)}) = \Delta(\alpha_0, \dots, \alpha_{n-1})^{\sigma'}$$

where  $\sigma'$  is defined by  $\sigma'(k) = \sum_{i=0}^k 2^i k_{\sigma^{-1}(i)}$ . One can check this fact by observing that  $d_{\sigma(i),n} \circ \sigma'(k) = (\sigma'(k))_{(\sigma(i))} = k_{(i)} = d_{i,n}(k)$ , and consequently

$$\begin{aligned} \Delta(\alpha_{\sigma(0)}, \dots, \alpha_{\sigma(n-1)}) &= \bigvee_{i < n} (\alpha_{\sigma(i)})^{d_{i,n}} = \bigvee_{i < n} (\alpha_{\sigma(i)})^{d_{\sigma(i),n} \circ \sigma'} \\ &= \bigvee_{i < n} ((\alpha_{\sigma(i)})^{d_{\sigma(i),n}})^{\sigma'} = \left( \bigvee_{i < n} (\alpha_i)^{d_{i,n}} \right)^{\sigma'} = \Delta(\alpha_0, \dots, \alpha_{n-1})^{\sigma'}. \end{aligned}$$

Finally,  $\sigma'(2^n - 1) = 2^n - 1$ , so the statement is true from Theorem 1.2 and Lemma 2.1.iii.  $\square$

**Proposition 5.2** (HC5, [1, Lemma 6.2]). *Let  $\mathbf{A}$  be a Mal'cev algebra and  $\alpha_0, \dots, \alpha_{n-1}, \gamma$  be congruences of  $\mathbf{A}$ . Then  $\alpha_0, \dots, \alpha_{n-2}$  centralizes  $\alpha_{n-1}$  modulo  $\gamma$  if and only if  $\gamma \geq [\alpha_0, \dots, \alpha_{n-1}]$ .*

*Proof.* The ‘only if’ part is given by the definition of the commutator, to prove the ‘if’ part suppose that  $\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-1})$  such that  $a_i \equiv_{\gamma} a_{i+2^{n-1}}$  for all  $i < 2^{n-1} - 1$ . We want to prove that  $a_{2^{n-1}-1} \equiv_{\gamma} a_{2^n-1}$ . By Lemma 4.2 we know that

$$a_{2^n-1} \equiv_{[\alpha_0, \dots, \alpha_{n-1}]} q_n(a_0, \dots, a_{2^n-2})$$

but the right hand side is modulo  $\gamma$  congruent to

$$q_n(a_0, \dots, a_{2^{n-1}-2}, a_{2^{n-1}-1}, a_0, \dots, a_{2^{n-1}-2}) = a_{2^{n-1}-1}.$$

So, we know that  $a_{2^{n-1}} \equiv a_{2^{n-1}-1}$  modulo  $[\alpha_0, \dots, \alpha_{n-1}] \vee \gamma = \gamma$ . And finally,  $\alpha_0, \dots, \alpha_{n-2}$  centralizes  $\alpha_{n-1}$  modulo  $\gamma$  from Lemma 3.3.  $\square$

The condition (HC6) is a direct corollary of condition (HC5); for a proof see [1, Corollary 6.3].

**Lemma 5.3.** *Let  $\mathbf{A}$  be an algebra,  $I$  a non-empty set,  $\rho_i \subseteq \text{Con } \mathbf{A}$  for all  $i \in I$ , and  $\alpha_0, \dots, \alpha_{n-2} \in \text{Con } \mathbf{A}$ . Then*

$$\Delta(\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in I} \rho_i) = \bigvee_{i \in I} \Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_i).$$

*Proof.* The statement can be derived directly from the definition of the relation  $\Delta(\alpha_0, \dots, \alpha_{n-1})$  by a simple calculation:

$$\begin{aligned} \Delta(\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in I} \rho_i) &= \left( \alpha_0^{d_{0,n}} \vee \dots \vee \alpha_{n-2}^{d_{n-2,n}} \vee \bigvee_{i \in I} \rho_i^{d_{n-1,n}} \right) \\ &= \bigvee_{i \in I} \left( \alpha_0^{d_{0,n}} \vee \dots \vee \alpha_{n-2}^{d_{n-2,n}} \vee \rho_i^{d_{n-1,n}} \right) \\ &= \bigvee_{i \in I} \Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_i). \end{aligned} \quad \square$$

The following lemma was in fact proved during the proof of [1, Lemma 6.4], but it is valid outside of the scope of Mal'cev algebras, and so far has not been stated so.

**Lemma 5.4.** *Let  $\mathbf{A}$  be an algebra,  $I$  a non-empty set,  $\rho_i \subseteq \text{Con } \mathbf{A}$  for all  $i \in I$ , and  $\alpha_0, \dots, \alpha_{n-2} \in \text{Con } \mathbf{A}$ . Then*

$$[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in I} \rho_i] = \bigvee_{\{i_0, \dots, i_{k-1}\} \subseteq I} [\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in \{i_0, \dots, i_{k-1}\}} \rho_i].$$

*Proof.* Throughout the proof we will extensively use the well-known fact that if  $\mathbf{A}_i$  for  $i \in J$  are subalgebras of some algebra  $\mathbf{A}$ , and  $a \in \bigvee_{i \in J} \mathbf{A}_i$  then there exists a finite set  $K \subseteq J$  such that  $a \in \bigvee_{i \in K} \mathbf{A}_i$ .

Let

$$\eta = \bigvee_{\{i_0, \dots, i_{k-1}\} \subseteq I} [\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in \{i_0, \dots, i_{k-1}\}} \rho_i].$$

First, we prove that  $\alpha_0, \dots, \alpha_{n-2}$  centralize  $\bigvee_{i \in I} \rho_i$  modulo  $\eta$ .

*Claim 1.* If  $\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in I} \rho_i)$  then there is a finite set  $S \subseteq I$  such that  $\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in S} \rho_i)$ .

The claim follows from Lemma 5.3 and the note at the beginning of this proof.

*Claim 2.* If  $a \equiv_\eta b$  then there is a finite set  $T \subseteq I$  such that  $a$  and  $b$  are congruent modulo  $[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in T} \rho_i]$ .

Again, there are finite sets  $T_0, \dots, T_{k-1}$  such that  $a$  and  $b$  are congruent modulo  $\bigvee_{j=0}^{k-1} [\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in T_j} \rho_i]$ . And,

$$\bigvee_{j=0}^{k-1} [\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in T_j} \rho_i] \leq [\alpha_0, \dots, \alpha_{n-2}, \bigvee_{\rho \in T} \rho]$$

where  $T = \bigcup_{j=0}^{k-1} T_j$ . Which completes the proof of the second claim.

Now, suppose that  $\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in I} \rho_i)$  and  $a_i \equiv_{\eta} a_{i+2^{n-1}}$  for all  $i < 2^{n-1} - 1$ . Let  $S$  be a finite set from Claim 1, and let  $T_i$  be finite sets such that  $a_i$  and  $a_{i+2^{n-1}}$  are congruent modulo  $[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{\rho \in T_i} \rho]$ ; such sets exist by Claim 2. Let  $U = S \cup T_0 \cup \dots \cup T_{2^{n-1}-2}$  (note that  $U$  is a finite set) and  $\eta' = \bigvee_{i \in U} \rho_i$ . Then

- (1)  $\mathbf{a} \in \Delta(\alpha_0, \dots, \alpha_{n-2}, \eta')$ , and
- (2)  $a_i \equiv_{[\alpha_0, \dots, \alpha_{n-1}, \eta']} a_{i+2^{n-1}}$  for all  $i < 2^{n-1} - 1$ .

So, from the Lemma 3.3 we know that  $a_{2^{n-1}-1} \equiv_{[\alpha_0, \dots, \alpha_{n-1}, \eta']} a_{2^n-1}$ . Finally,  $[\alpha_0, \dots, \alpha_{n-1}, \eta'] \leq [\alpha_0, \dots, \alpha_{n-1}, \eta]$  because  $\eta' \leq \eta$ . Hence  $\alpha_0, \dots, \alpha_{n-2}$  centralize  $\bigvee_{i \in I} \rho_i$  modulo  $\eta$ , and consequently  $[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in I} \rho_i] \leq \eta$ .

The other inclusion is obvious from the fact that

$$[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in I} \rho_i] \geq [\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in J} \rho_i]$$

for every finite set  $J \subseteq I$ . □

**Lemma 5.5** ([1, Corollary 6.6]). *Let  $\mathbf{A}$  be a Mal'cev algebra and  $\alpha_1, \dots, \alpha_{n-1}, \rho_1, \dots, \rho_k$  congruences of  $\mathbf{A}$ . Then*

$$[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i=1}^k \rho_i] = \bigvee_{i=1}^k [\alpha_0, \dots, \alpha_{n-2}, \rho_i].$$

*Proof.* It suffices to prove the statement just for  $k = 2$ . We will write

$$(a_0, \dots, a_{2^{n-1}-1}) \equiv_{\Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_i)} (b_0, \dots, b_{2^{n-1}-1})$$

if  $(a_0, \dots, a_{2^{n-1}-1}, b_0, \dots, b_{2^{n-1}-1}) \in \Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_i)$ . Note that from Lemma 3.4, we know that the binary relation  $\{(\mathbf{a}, \mathbf{b}) : \mathbf{a} \equiv_{\Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_i)} \mathbf{b}\}$  is a congruence on  $\Delta(\alpha_0, \dots, \alpha_{n-2})$ . From Lemma 5.3, we know that

$$\Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_1 \vee \rho_2) = \Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_1) \vee \Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_2).$$

Since in Mal'cev algebras  $\alpha \circ \beta = \text{Sg}(\alpha \cup \beta)$  for any pair of congruences  $\alpha, \beta$ , we have that for all  $\mathbf{a}, \mathbf{b} \in \Delta(\alpha_0, \dots, \alpha_{n-2})$ ,  $\mathbf{a} \equiv_{\Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_1 \vee \rho_2)} \mathbf{b}$  if and only if there exists  $\mathbf{c}$  such that  $\mathbf{a} \equiv_{\Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_1)} \mathbf{c}$  and  $\mathbf{b} \equiv_{\Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_2)} \mathbf{c}$ .

Now, we prove that

$$(5.1) \quad [\alpha_0, \dots, \alpha_{n-2}, \rho_1 \vee \rho_2] \leq [\alpha_0, \dots, \alpha_{n-2}, \rho_1] \vee [\alpha_0, \dots, \alpha_{n-2}, \rho_2].$$

Suppose that  $a$  and  $b$  are congruent modulo  $[\alpha_0, \dots, \alpha_{n-2}, \rho_1 \vee \rho_2]$ , hence from Proposition 3.6 there are  $e_0, \dots, e_{2^{n-1}-2}$  such that

$$(e_0, \dots, e_{2^{n-1}-2}, a) \equiv_{\Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_1 \vee \rho_2)} (e_0, \dots, e_{2^{n-1}-2}, b).$$

From the above observation, we know that there is a tuple  $\mathbf{c} \in \Delta(\alpha_0, \dots, \alpha_{n-2})$  such that

$$(5.2) \quad \mathbf{c} \equiv_{\Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_1)} (e_0, \dots, e_{2^{n-1}-2}, a)$$

and

$$(5.3) \quad \mathbf{c} \equiv_{\Delta(\alpha_0, \dots, \alpha_{n-2}, \rho_2)} (e_0, \dots, e_{2^{n-1}-2}, b)$$

If we use Lemma 4.2 for (5.2), we get that

$$a \equiv_{[\alpha_0, \dots, \alpha_{n-2}, \rho_1]} q_n(c_0, \dots, c_{2^{n-1}-1}, e_0, \dots, e_{2^{n-1}-2});$$

similarly for (5.3), we get that

$$b \equiv_{[\alpha_0, \dots, \alpha_{n-2}, \rho_2]} q_n(c_0, \dots, c_{2^{n-1}-1}, e_0, \dots, e_{2^{n-1}-2}).$$

Altogether,  $a$  and  $b$  are congruent modulo  $[\alpha_0, \dots, \alpha_{n-2}, \rho_1] \vee [\alpha_0, \dots, \alpha_{n-2}, \rho_2]$ . Which completes the proof of (5.1). The other inclusion is given by (HC2).  $\square$

**Proposition 5.6** (HC7, [1, Lemma 6.7]). *Let  $\mathbf{A}$  be a Mal'cev algebra with congruences  $\alpha_0, \dots, \alpha_{n-2}$ , and  $\rho_i$ ,  $i \in I$  for  $I$  non-empty set. Then*

$$[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in I} \rho_i] = \bigvee_{i \in I} [\alpha_0, \dots, \alpha_{n-2}, \rho_i].$$

*Proof.* If  $I$  is a finite set then the proposition is given by Lemma 5.5. If  $I$  is infinite, we can first use Lemma 5.4 to get

$$[\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in I} \rho_i] = \bigvee_{\{i_0, \dots, i_{k-1}\} \subseteq I} [\alpha_0, \dots, \alpha_{n-2}, \bigvee_{i \in \{i_0, \dots, i_{k-1}\}} \rho_i].$$

Then by using the finite case, the right hand side is equal to

$$\bigvee_{\substack{\{i_0, \dots, i_{k-1}\} \subseteq I, \\ i \in \{i_0, \dots, i_{k-1}\}}} [\alpha_0, \dots, \alpha_{n-2}, \rho_i] = \bigvee_{i \in I} [\alpha_0, \dots, \alpha_{n-2}, \rho_i]. \quad \square$$

**Proposition 5.7** (HC8, [1, Proposition 6.14]). *Let  $\mathbf{A}$  be a Mal'cev algebra with congruences  $\alpha_0, \dots, \alpha_{n-1}$ , and  $i \in \{1, \dots, n-1\}$ . Then*

$$[[\alpha_0, \dots, \alpha_{i-1}], \alpha_i, \dots, \alpha_{n-1}] \leq [\alpha_0, \dots, \alpha_{n-1}].$$

*Proof.* For  $m \geq i$ , define the map  $e_m: \{0, \dots, 2^m - 1\} \rightarrow \{0, \dots, 2^{m-i} - 1\}$  by

$$k \mapsto k_{(0)} \cdots k_{(i-1)} + \sum_{j < m-i} 2^{j+1} k_{(j+i)}.$$

We claim that

$$(5.4) \quad \Delta([\alpha_0, \dots, \alpha_{i-1}], \alpha_i, \dots, \alpha_{m-1})^{e_m} \leq \Delta(\alpha_0, \dots, \alpha_{m-1}).$$

Because  $\Delta([\alpha_0, \dots, \alpha_{i-1}], \alpha_i, \dots, \alpha_{m-1})^{e_m}$  is clearly a subuniverse of  $\mathbf{A}^{2^n}$  generated by the set

$$([\alpha_0, \dots, \alpha_{i-1}]^{d_{0,m-i+1}})^{e_m} \cup \bigcup_{j < m-i} (\alpha_{i+j}^{d_{j+1,m-i+1}})^{e_m},$$

it suffices to prove that this is a subset of  $\Delta(\alpha_0, \dots, \alpha_{n-1})$ . The inclusions

$$(\alpha_{i+j}^{d_{j+1,m-i+1}})^{e_m} \subseteq \Delta(\alpha_0, \dots, \alpha_{n-1})$$

are consequences of the fact that  $d_{j+1,m-i+1} \circ e_m = d_{j+i,m}$  for all  $j < m-i$ . The other inclusion,  $([\alpha_0, \dots, \alpha_{i-1}]^{d_{0,m-i+1}})^{e_m} \subseteq \Delta(\alpha_0, \dots, \alpha_{n-1})$ , can be proved by induction on  $m$ . If  $m = i$  then

$$([\alpha_0, \dots, \alpha_{i-1}]^{d_{0,1}})^{e_m} = \{(a, \dots, a, b) \mid a \equiv_{[\alpha_0, \dots, \alpha_{i-1}]} b\},$$

and all the elements of this set are in  $\Delta(\alpha_0, \dots, \alpha_{i-1})$  from Proposition 3.6. For the induction step, observe that

$$\pi_{\{k_{(m)}=0\}}((a, b)^{e_{m+1}}) = \pi_{\{k_{(m)}=1\}}((a, b)^{e_{m+1}}) = (a, b)^{e_m},$$

so the inclusion follows from Lemma 3.4.ii.

Finally, from (5.4) for  $m = n$ , the fact that  $e_n(k) = 2^{n-i+1} - 1$  if and only if  $k = 2^n - 1$ , and Lemma 2.1 we get

$$\psi_{2^{n-i+1}-1}(\Delta([\alpha_0, \dots, \alpha_{i-1}], \alpha_i, \dots, \alpha_{n-1})) \leq \psi_{2^n-1}(\Delta(\alpha_0, \dots, \alpha_{n-1})).$$

The rest is Theorem 1.2. □

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